THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Problem Set 7

1. Let $f(x,y) = (f_1(x,y), f_2(x,y)) = (\sqrt{xy}, \sqrt{\frac{y}{x}})$.

   (a) Find the Jacobi matrix $J_f(x,y)$ and evaluate it at the point $(x,y) = (2,8)$.

   (b) By using the linearization of the function f at the point $(x,y) = (2,8)$, approximate $f(1.9, 8.2)$.

2. Express $\frac{dw}{dt}$ as a function of t if

   (a) $w = x^2 + 2xy$, $x = \cos 2t$, $y = \sin 3t$;

   (b) $w = \ln(xy + yz + zx)$, $x = t^2$, $y = e^t$, $z = \cos t$;

3. Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of u and v if

   (a) $z = 3e^{2x} \ln y$, $x = \ln(u+v)$, $y = uv$;

   (b) $z = xe^y + ye^x$, $x = u + v$, $y = \ln x$.

4. If $f(u,v,w)$ is differentiable and $u = x - y$, $v = y - z$ and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

5. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathbb{R}^3$ and let $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

   Show that for any positive integer n,

$$\nabla(r^n) = nr^{n-2}\mathbf{r}.$$

6. A function $f(x,y)$ is said to be a **harmonic** if it satisfies the **Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

   For $(x,y) \neq (0,0)$, f can be regarded as a function of r and θ with $r > 0$ and $0 \leq \theta < 2\pi$ by

$$f(r,\theta) = f(x(r,\theta), y(r,\theta)),$$

   where $x(r,\theta) = r\cos\theta$, $y(r,\theta) = r\sin\theta$ and (r,θ) is called the polar coordinates.

   Show that the Laplace equation in polar coordinates can be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0.$$